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Derivation of the Reach Time Equation—Based on the Arrhenius Equation, a Theoretical Equation Describing the Time Distribution to Reach a Certain Stage, such as Germination, in a Population

Yoshitaka HARA*

Department of Research Promotion, Kyushu-Okinawa Agricultural Research Center, National Agriculture and Food Research Organization, Koshi, Japan

Abstract

The reach time equation, a sigmoidal equation with the same number of parameters as the logistic equation, is derived by adding a concentration factor to the physically theoretical Arrhenius equation and assuming a normally distributed factor. Because this equation provides a good description of the time distribution to germination, it may be applicable as an equation for the time distribution reaching a certain stage in a population. Moreover, as derived from the Arrhenius equation, this equation can also be used to evaluate the effect of temperature. Using this equation, the skewness in the time distribution reaching a certain stage became proportional to the variation coefficient of the concentration factor, indicating that a high variation coefficient for the amount of relevant substrates, such as nutrients and enzymes, caused the skewness in the distribution. Given the theoretical implications of the parameters obtained by fitting this equation, it is predicted that the growth and development of living organisms, including inorganic changes can be theoretically analyzed by fitting and analyzing the parameter values. In addition, owing to the small number of parameters, it will also be useful for simple fitting as a sigmoidal curve.

Discipline: Crop Science

Additional key words: logistic equation, rice, Richards equation, seeds, weed

Introduction

Temporal trends in the rate at which individuals reach a certain stage can be described in terms of the cumulative distribution of the rate at which they reach that stage over time. Sigmoidal growth curves are used to describe cumulative distributions such as seed germination rates (Hara 1999, 2001, 2005) and nutrient supply from controlled-release fertilizers (Hara 2000, 2001). The logistic equation is a typical growth equation, which describes a growth curve describing a symmetrical distribution without skewness before and after the median in the original frequency distribution of the cumulative distribution. The Gompertz equation describes an asymmetric distribution, and the Richards equation encompasses both distribution types and can describe a range of different distributions (Hara 1999, 2001). These equations are theoretical in describing the following characteristics regarding the size of an individual or population of a given organism. The initial growth stage is approximately exponential, then assumes a linear pattern approaching saturation, and eventually plateaus on reaching maturity (Wikipedia 2023b). However, these equations are not theoretical in describing the cumulative distribution of the rate of individuals reaching a certain stage in a population, such as seed germination or nutrient supply from controlled-release fertilizers, and are often used as they provide a good fit.

Shibuya & Hayashi (1984) proposed a theoretical equation based on a model with several fluctuation events before germination occurred. However, although this equation is simple with a small number of parameters, it is limited because the parameter, the

*Corresponding author: yshara@affrc.go.jp

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number of fluctuations, is limited to integers, making it difficult to represent small differences in the distribution. Therefore, in this study, the author proposed new equations derived from a different model.

The effect of temperature on the time taken to reach a certain stage can be described using the physically theoretical Arrhenius equation (Wikipedia 2023a). For example, the time from sowing to germination or the time taken for nutrients to be supplied from controlled-release fertilizers after application has been evaluated for the degree of temperature effect using the parameter activation energy, which is obtained by fitting the Arrhenius equation for the change in a representative population value, such as the median, with respect to temperature (Hara 2000, 2001, 2005). There have been a few cases in which the effect of temperature has been examined for the entire distribution, although the distribution is not necessarily the same even if the median and other representative values are the same.

In this study, a new theoretical equation was derived to describe the time distribution required to reach a certain stage based on the Arrhenius equation, which has a theoretical basis in physics. In contrast to the previously applied growth equations, this equation is theoretical, and thus, it is possible to obtain values for theoretical parameters. Moreover, it is practical because it can represent a skewed distribution based on a small number of parameters. This study describes the utility and potential applications of this equation using seed germination data.

Theory

1. Derivation of the reach time equation

The Arrhenius equation, which has a physics basis, is as follows (Wikipedia 2023a):

$$k = A \cdot exp(-E/R/T), \qquad \text{Eq. 1}$$

where A is the frequency factor, E is the activation energy, and R is the gas constant.

When limited to elementary reactions, the rate constant (k) can be obtained by removing the effect of the reactant concentration (c; hereafter referred to as the concentration factor) from the rate of the reaction (r) as follows (Wikipedia 2023c):

$$r = c \cdot k \qquad \text{Eq. 2}$$

As the rate is the reciprocal of time, the time required for a reaction can be obtained as follows:

$$t = 1/r \qquad \text{Eq. 3}$$

From Eq. 2 and Eq. 3, the following equation is obtained:

$$t = 1/(c \cdot k) \qquad \text{Eq. 4}$$

Substituting Eq. 1 into Eq. 4 yields the following equation:

$$t = exp(E/R/T) \,/\, (A \cdot c) \qquad \text{Eq. 5}$$

In a reaction system comprising several reactions, the time required to complete the entire reaction sequence is the sum of the times required for each reaction. If the reaction system has the slowest reaction (rate-limiting reaction), the proportion of time for the rate-limiting reaction in the sum will be higher, and the effect of temperature in the sum reflects the effect of temperature on the rate-limiting reaction. Accordingly, even for a system consisting of several reactions, Eq. 5 is expected to hold if it could be regarded as a single reaction.

Thus, it is possible to consider applying this idea to more complex reactions such as those involved in seed germination. Even in the case of uniform seeds, not all seeds germinate simultaneously, with some seeds germinating early and others late. The number of seeds germinating with respect to the time elapsed since sowing varies with the germination curve characterized by a period of high germination in the center and lower rates at either tail of the curve.

If Eq. 5 follows this characterization, then the right-hand side should be correspondingly distributed because t on the left-hand side is distributed in the seed population. On the right side, the gas constant (R) is constant, activation energy (E) and frequency factor (A) are specific to the type of reaction and are considered to be constant, and the absolute temperature (T; hereafter simply referred to as the temperature) does not differ among the populations. However, when germination is considered as a reaction system, the concentration factor (c) of the reactants can represent the amounts of reactants, such as nutrients and enzymes, and these amounts are predicted to vary within the population.

Accordingly, given the variable nature of the concentration factor (c), it is assumed to be normally distributed (mean μ_c, standard deviation σ_c). Eq. 5 shows that the smaller the value of c, the larger the value of t and the slower the germination. This indicates that when seeds with a large concentration factor (c) germinate, the germination rate (y), which is the percentage of germinated seeds in the population, is still small, which can be expressed as follows:

$$c = CNI(100\% - y, \mu_c, \sigma_c), \qquad \text{Eq. 6}$$

where *CNI* represents the inverse function of the normal cumulative distribution function, which provides the value of c when the cumulative probability is $100\% - y$ in a normal distribution with a mean value μ_c and a standard deviation value σ_c. Whereas this cannot be expressed using a simple formula, it can be calculated using a spreadsheet function (e.g., norm.inv of Microsoft Excel).

The frequency factor (A) is considered to take a value specific to the reaction, but the details are unknown and there are few opportunities to use it. Thus, instead of using Eq 6, the distribution of $A \cdot c$ (mean μ, standard deviation σ) was used as follows:

$$A \cdot c = CNI(100\% - y, \mu, \sigma) \qquad \text{Eq. 7}$$

Substituting Eq. 7 into Eq. 5 yields Eq. 8 as follows:

$$t = exp(E/R/T) \,/\, CNI(100\% - y, \mu, \sigma) \qquad \text{Eq. 8}$$

Solving for y yields the following equation:

$$y = 100\% - CN[exp(E/R/T) \,/\, t, \mu, \sigma], \qquad \text{Eq. 9}$$

where *CN* represents the normal cumulative distribution function, which provides the cumulative probability for a given value $[exp\ (E/R/T)/t]$ in a normal distribution with mean μ and standard deviation σ. As with Eq. 6, this cannot be expressed in a simple formula but can be calculated using a spreadsheet function (e.g., norm.dist of Microsoft Excel).

Whereas Eq. 9 can be applied when the final rate is 100%, and there are cases in which the final rate is not 100% owing to factors such as premature mortality. To account for this outcome, the final rate (F) was inserted as follows:

$$y = F \cdot \{100\% - CN[exp(E/R/T) \,/\, t, \mu, \sigma]\} \qquad \text{Eq. 10}$$

This equation is derived by assuming that the concentration factor c is normally distributed; however, different equations can be derived using distributions other than a normal distribution. Thus, various distributions can be introduced instead of the normal distributions. Since it is not essential that the distribution be normal, Eq. 10 can be referred to as the equation describing the cumulative distribution of reach times, or more concisely, the “reach time equation,” without including the “normal distribution.”

The reach time equation describes the cumulative distribution of the time required to reach a certain stage in a population, as well as the temporal trend of the frequency (or percentage) of reaching a certain stage in a population. It shows the relationship between the time (t) and reach rate (y), and requires the following five parameters: the final rate (F), activation energy (E), temperature (T), mean (μ) and standard deviation (σ) of the distribution of $A \cdot c$.

As with the Arrhenius equation, in this reach time equation, a single reaction is assumed. If it is desirable to assume multiple rate-limiting reactions, as in the case of germination at low temperatures as shown in Hara (2005), the right hand side of Eq. 8 should be summed over the number of reactions. For example, if two rate-limiting reactions α and β are assumed, the equation takes the following form:

$$t = exp(E_\alpha/R/T) \,/\, CNI(100\% - y/F, \mu_\alpha, \sigma_\alpha) + exp(E_\beta/R/T) \,/\, CNI(100\% - y/F, \mu_\beta, \sigma_\beta) \qquad \text{Eq. 11}$$

In this case, because the germination rate (y) is difficult to determine analytically, it should be calculated numerically.

2. Parameter reduction of the reach time equation

Because the shape of the cumulative frequency distribution is the same regardless of the scale of the time axis, the following equation holds:

$$CN[exp(E/R/T)/t, \mu, \sigma] = CN[1/t, \mu/exp(E/R/T), \sigma/exp(E/R/T)] \qquad \text{Eq. 12}$$

If temperature (T) is not used as a parameter, Eq. 15 can be derived from Eq. 10 by substituting Eqs. 13 and 14, as follows:

$$p = \mu/exp(E/R/T) \qquad \text{Eq. 13}$$

$$q = \sigma/exp(E/R/T) \qquad \text{Eq. 14}$$

$$y = F \cdot [100\% - CN(1/t, p, q)], \qquad \text{Eq. 15}$$

for which three parameters are required (F, p, and q).

Accordingly, when the distinction is necessary, Eq. 10 is referred to as the five-parameter-type reach time equation and Eq. 15 as the three-parameter-type reach time equation.

3. Differentiation of the reach time equation

As previously mentioned, although it is impossible to describe a normal cumulative distribution with a simple formula, a normal distribution can be described using a formula. When $A \cdot c$ is normally distributed with the mean (μ) and standard deviation (σ), the probability density [$f(A \cdot c)$] is expressed as follows:

$$f(A \cdot c) = \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{(A \cdot c - \mu)^2}{2\sigma^2}\right] \qquad \text{Eq. 16}$$

The derivative (dy/dt) of the germination rate (y) with time (t) was obtained using Eq. 10 as follows:

$$\frac{dy}{dt} = \frac{F \cdot exp(E/R/T)}{\sqrt{2\pi}\sigma \cdot t^2} \cdot exp\left\{-\frac{[exp(E/R/T)/t - \mu]^2}{2\sigma^2}\right\} \qquad \text{Eq. 17}$$

This provides the distribution (y) of individuals (e.g., seeds) reaching a certain stage (e.g., germination) at time (t) and temperature (T), and the number of parameters required is five (F, E, T, μ, and σ). To differentiate between Eqs. 10 and 17, the former is referred to as an integral-type reach time equation and the latter will be described as a differential-type reach time equation.

As in the case of the integral type, if the temperature (T) is not used as a parameter, Eq. 18 can be derived from Eq. 17 by substituting into with Eqs. 13 and 14 as follows:

$$\frac{dy}{dt} = \frac{F}{\sqrt{2\pi} \cdot q \cdot t^2} \cdot exp\left\{-\frac{[1/t - p]^2}{2 \cdot q^2}\right\}, \qquad \text{Eq. 18}$$

for which three parameters are required (F, p, and q).

Thus, when the distinction is necessary, Eq. 17 will be referred to as the five-parameter-type differential type of the reach time equation, and Eq. 18 is referred to as the three-parameter-type differential type. Accordingly, along with Eqs. 10 and 15, four types of reach time equations are available: (integral type/differential type) × (5-parameter type/3-parameter type).

As previously mentioned, the reach time equation is a theoretical equation based on the Arrhenius equation, which has a temperature parameter and can be readily calculated using spreadsheet functions, the differential type of which can be described using a formula. Furthermore, if the temperature parameter is not required, the number of parameters can be reduced to three, which is the same as the number of parameters used for the logistic equation, a growth equation that has been used to describe the curve of the germination rate over time. Before the present study, there had been no simple theoretical equations applicable to express the distribution of reach times (or trends in the number or percentage reached over time) of a population, and it is thus anticipated that this equation will find practical application in describing and theoretically analyzing such reach times.

4. Characteristics of the curve described by the reach time equation

As previously mentioned, when using a growth equation, such as the logistic or Richards equation, it is impossible to theoretically describe the cumulative distribution of individuals that have reached a certain stage in a population. However, the reach time equation is based on the assumption that the concentration factor in the Arrhenius equation is normally distributed. By including time (t) in the reciprocal, the reach time equation can be used to describe skewed distribution. Similar to the logistic equation, the three-parameter type of reach time equation (Eq. 15) does not include the temperature (T). The four-parameter Richards equation also excludes the temperature parameter (T) and can be used to describe a skewed distribution. Consequently, the reach time and Richards equations were contrasted as follows.

To gain a better understanding of the characteristics of the time distribution in a population, such as germination, Hara (1999, 2001) derived four population parameters: percentage at infinity (Vi), median of time (Me), quartile deviation of time (Qu), and quartile skewness of time (Sq). Notably, in the first of these papers, Hara (1999, 2001) defined skewness as Sk but subsequently changed it to Sq in the second publication, as it is mathematically general. Vi has the same meaning as the final rate (F) in the reach time equation, and thus for convenience, let $Vi = F = 100\%$. The other parameters of the Richards equation can be reversibly transformed to yield three other population parameters (Me, Qu, and Sq). Accordingly, one relational equation can be obtained by obtaining Me, Qu, and Sq for the three-parameter type of the reach time equation with one fewer parameter than the Richards equation.

The mean (μ) is the median, the surrounding distributions of which are symmetrical, as $A \cdot c$ is normally distributed, as in Eq. 7. When the order in which the parameters are sorted from smallest to largest in the distribution of each parameter is indicated by a subscript for each parameter as a percentage of the total number, the following definition can be made: the first, second (median), and third quartiles of $A \cdot c$ are $A \cdot c_{25\%} = \mu - h$, $A \cdot c_{50\%} = \mu$, and $A \cdot c_{75\%} = \mu + h$, respectively. Frequency factor (A) is considered dependent on the reaction system and does not vary with germination time, whereas concentration factor (c) may be related to factors of interest, such as the amount of nutrients or enzymes that would affect the reaction. The larger the factor (c) and $A \cdot c$, the faster the expected rate of germination. Thus, the germination time for seeds of $A \cdot c_{75\%}$, $A \cdot c_{50\%}$, and $A \cdot c_{25\%}$ would be $t_{25\%}$, $t_{50\%}$, and $t_{75\%}$, respectively, derived as follows:

$$t_{25\%} = exp(E/R/T) \,/\, (A \cdot c_{75\%}) = exp(E/R/T) \,/\, (\mu + \hbar) \quad \text{Eq. 19}$$

$$t_{50\%} = exp(E/R/T) \,/\, (A \cdot c_{50\%}) = exp(E/R/T) \,/\, \mu \quad \text{Eq. 20}$$

$$t_{75\%} = exp(E/R/T) \,/\, (A \cdot c_{25\%}) = exp(E/R/T) \,/\, (\mu - \hbar) \quad \text{Eq. 21}$$

As $A{\cdot}c$ is normally distributed, it can be calculated as follows:

$$h = A \cdot c_{75\%} - \mu = CNI(75\%, \mu, \sigma) - \mu = CNI(75\%, 0, 1) \cdot \sigma \quad \text{Eq. 22}$$

The population parameters for germination time (Hara 2001) were obtained as follows:

$$Me = t_{50\%} = exp(E/R/T) \,/\, \mu \quad \text{Eq. 23}$$

$$Qu = (t_{75\%} - t_{25\%}) \,/\, 2 = exp(E/R/T) \cdot \frac{CNI(75\%,0,1)\cdot\sigma}{\mu^2 - [CNI(75\%,0,1)\cdot\sigma]^2} \quad \text{Eq. 24}$$

$$Sq = \frac{(t_{75\%} - t_{50\%}) - (t_{50\%} - t_{25\%})}{t_{75\%} - t_{25\%}} = \hbar/\mu = CNI(75\%, 0, 1) \cdot \sigma/\mu \cong 0.675 \cdot \sigma/\mu \quad \text{Eq. 25}$$

Accordingly, whereas *Me* and *Qu* are affected by temperature, *Sq* is unaffected and proportional to the variation coefficient (CV = σ/μ) of $A{\cdot}c$. The actual distribution of cumulative relative frequencies for different variation coefficients is shown in Figure 1, showing that the distributions before and after the median, fixed at 1.0 for clarity, are closer to symmetry for smaller variation coefficients, and the tail of the distribution after the median extends for larger variation coefficients.

Furthermore, from Eqs. 23, 24, and 25, the following equation is obtained:

$$Qu/Me = Sq \,/\, (1 - Sq^2), \quad \text{Eq. 26}$$

which shows that a single relational equation holds among *Me*, *Qu*, and *Sq*, regardless of the temperature and other parameters.

This relationship is consistent with the fact that the number of parameters in the three-parameter type of reach time equation (Eq. 15) without the temperature parameter is one less than that in the Richards equation. This relationship may indicate a characteristic of the distribution of reach time to a certain stage, attributable to the distribution of $A{\cdot}c$, and of the reach time equation.

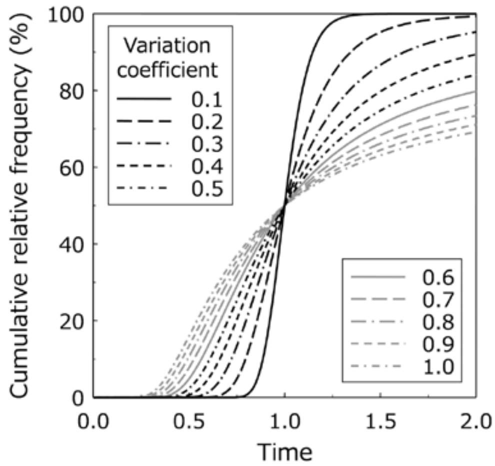

Fig. 1. Effects of the variation coefficient of $A{\cdot}c$ on the shape of the reach time equation curve
Assuming that $A{\cdot}c$ is normally distributed with a mean of 1 and different variation coefficients, the curve was calculated from the reach time equation.

Materials and methods

1. Fitting the reach time equation to the trend in germination rate

There are two situations in which the reach time equation can be applied. The first involves an analysis that excludes the effect of temperature, using the three-parameter-type reach time equation (Eq. 15 for the integral type or Eq. 18 for the differential type), whereas the second involves an analysis that includes the effect of temperature, using the five-parameter-type equation (Eq. 10 for the integral type or Eq. 17 for the differential type).

Analysis excluding the effect of temperature is comparable to cases in which the logistic and Richards equations have been used, and because it is important to simply fit with a small number of parameters, it is realistic to use a simpler type of reach time equation. To verify this assumption, the three-parameter-type reach time equation (Eq. 15) was fitted to the data of trends in the rate of rice seed germination under the 32 conditions [16 temperatures × 2 seed lots (Lot +N, −N)] as used by Hara (2005).

With respect to analyses in which the effects of temperature are considered, simplicity and good fit are important. In this regard, Hara (2005) reported that it is desirable to assume two rate-limiting reactions in the temperature range 10°C-20°C. Thus, to verify this case, the five-parameter-type reach time equations for the two reactions (Eq. 11) were fitted to the aforementioned data.

For comparison, fitting was also performed using the Richards equation.

In this study, all fittings were performed using a spreadsheet software (Excel, Microsoft Corp.).

2. Relationships obtained from the reach time equation for the trend in germination rate

Using the reach time equation, the relationship in Eq. 26 is obtained. It was examined whether this relationship actually holds for the distribution of germination time, an example of the time distribution required to reach a certain stage. Specifically, three population parameters (*Me*, *Qu*, and *Sq*) were calculated by fitting the Richards equation to the distribution of rice germination time under the aforementioned 32 conditions (Hara 2005) and scatter plots of *Qu*/*Me* and *Sq* were drawn.

3. Skewness of germination time distribution in plant species

The skewness in the distribution of the reach time is proportional to the coefficient variation of $A{\cdot}c$ (Eq. 25). To aid interpretation, the coefficient variation (σ/μ) of $A{\cdot}c$ was obtained by applying the three-parameter-type reach time equation (Eq. 15) on the germination rate trends of two weed species (*Digitaria adscendens*: mehishiba in Japanese and *Chenopodium album:* shiroza in Japanese) and paddy rice (*Oryza sativa*). For the two weed species, the trends in germination rate at 25°C were obtained from Shibuya & Hayashi (1984), whereas similar data for paddy rice were obtained from Hara (2005) for one arbitrarily selected case (Lot +N at 16.5°C).

Results

1. Fitting the reach time equation to the trend in germination rate

The fit was good when the three-parameter-type reach time equation was fitted under each assessed condition. For example, eight results for half of the population (Lot +N) are shown in Figure 2(a). The average error for all the measurement points was 1.6% of the number of seeds used. In this case, 96 parameters were used in the fitting (= 3 parameters × 32 conditions).

When the serial five-parameter type of two reach time equations for two reactions were fitted under all the conditions, the fit was sufficient, for which eight results for half of the population (Lot +N) are shown in Figure 2(b) as an example. The average error for all measurement points was 4.1%, and the number of parameters used was 12 {= [1 (E) + 2 (μ, σ) × 2 seed lots] × 2 reactions + 1 (F) × 2 seed lots}.

For comparison, when the Richards equation was applied to the same germination rate data under each condition, the average error for all measurement points was 1.3% and 128 parameters were used (= 4 parameters × 32 conditions). When the serial equation comprising two Richards equations for two reactions was fitted under all conditions, the average error for all measurement points was 4.1% and the number of parameters used was 16 {= [1 (E) + 3 parameters × 2 seed lots] × 2 reactions + 1 parameter × 2 seed lots}.

Compared to the Richards equation, fitting using the reach time equation did not contribute to any appreciable increase in the average error, even when using a smaller number of parameters. This indicates that the use of reach time equation is relevant and practical.

2. Relationships obtained from the reach time equation for the trend in germination rate

Plots of *Qu*/*Me* and *Sq* for 32 conditions (= 16 temperatures × 2 seed lots) reveal a wide distribution of data points above and below the fitted curve, indicating the relationship in Eq. 26 (Fig. 3).

Hara (1999) reported the number of measurements required to obtain a 10% variance coefficient at a 95% confidence level for population parameters, namely, 0.1 for *Me*, 4 for *Qu*, and more than 1,000 for *Sk* [= (*Me* − mode)/*Qu*]. It is anticipated that in common *Sk*, *Sq*, proposed as an alternative to *Sk* (Hara 2000), will be more prone to variation, although this is not shown here.

As these parameters were measured only a once for each condition, it is anticipated that *Qu*, and particularly *Sq*, will contain large errors, possibly resulting in data points being distributed at a distance from the curve. The fact that the curve passes through the midpoint of the scatter area of the plot may not rule out conformation to the relationship, but rather indicates the possibility of conformation as the plots should show equal amounts of up and down scatter despite the errors.

3. Skewness of germination time distribution in plant species

The reach time equation was showed a good fit for trends in the germination rates of *D. adscendens*, *C. album*, and *O. sativa*, with mean errors of 0.87%, 0.73%, and 1.96%, respectively (Fig. 4). The variation coefficients (σ/μ) of $A{\cdot}c$ obtained from the fitted equations for *D. adscendens*, *C. album*, and *O. sativa* were 0.51, 0.31, and 0.14, respectively, whereas the corresponding values of *Sq*, which is proportional to σ/μ (Eq. 25), were 0.34, 0.21, and 0.095. Compared with that

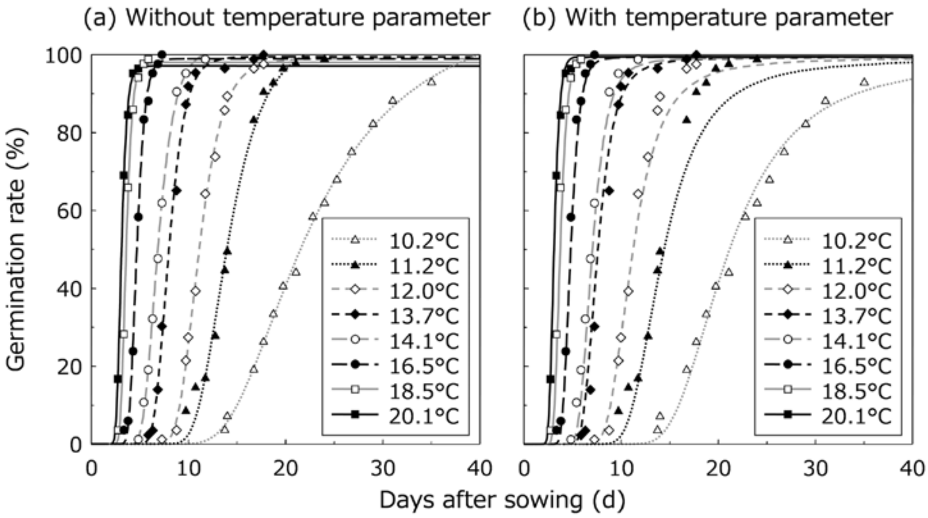

Fig. 2. Fitting of the reach time equation to the trend in germination rate, without/with inclusion of the temperature parameter

For trends in the germination rate of rice seeds at different temperatures for the same seed population, the measured values are shown as symbols, and the calculated values are shown as curves. Data were fitted with a different reach time equation (three-parameter type without the temperature parameter) at different temperatures (a) or fitted using a serial equation comprising two reach time equations (five-parameter type with the temperature parameter) for the two reactions at all temperatures (b).

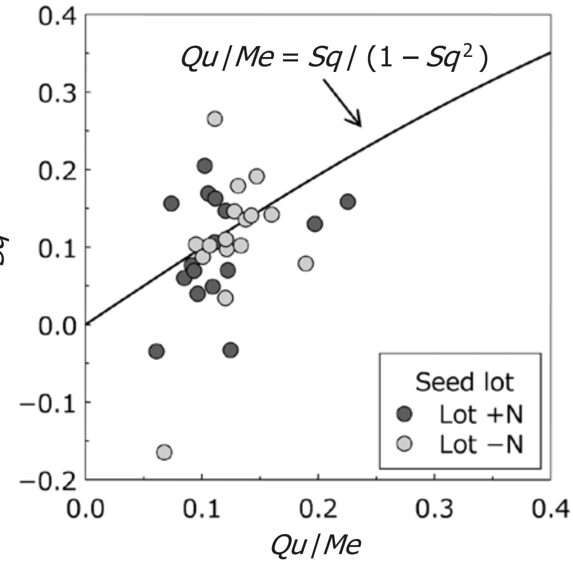

Fig. 3. Relationship established using the reach time equation

For the relationship between [quartile deviation (*Qu*)/ median (*Me*)] and quartile skewness (*Sq*), the results of the distribution of germination times obtained for each of the 16 temperature conditions for the two rice seed populations (denoted by different symbols) and the relationship established using the reach time equation (denoted by a curve) are shown.

of *O. sativa*, the shape of the germination time distribution was more backward-skewed for the two weed species.

The observed variation in $A{\cdot}c$ can be attributed to variations in either the frequency factor (A) or concentration factor (c). The value of the former should be specific to the type of germination response, whereas the latter should represent the concentration or amount of nutrients, enzymes, and other substances associated with germination. It is reasonable to assume that the reactions contributing to germination are similar, even among different species, whereas the concentration of factors associated with germination is generally population-specific. Accordingly, it can be assumed that the variation in $A{\cdot}c$ is more likely to be attributable to the variation in the concentration factor (c) than to that in the frequency factor (A). Thus, the skewness of the germination time distribution, associated with the variation coefficients of $A{\cdot}c$ obtained from the fitted reach time equations, can be ascribed to the variations in the respective concentration factors.

As a cultivated crop, rice has a higher seed weight (ca. 24 mg in Hara & Toriyama (1998)) than the smaller-seeded weed species (ca. 0.7 mg for *D. adscendens* and *C. album* in Ito (1993)). Consequently, the variation coefficients of the concentration factors associated with rice populations may be relatively small, whereas those associated with weed populations may be relatively large. This may account for the large coefficient of variations obtained for the two weed species and the relatively small value obtained for paddy rice, although other possibilities should not be excluded.

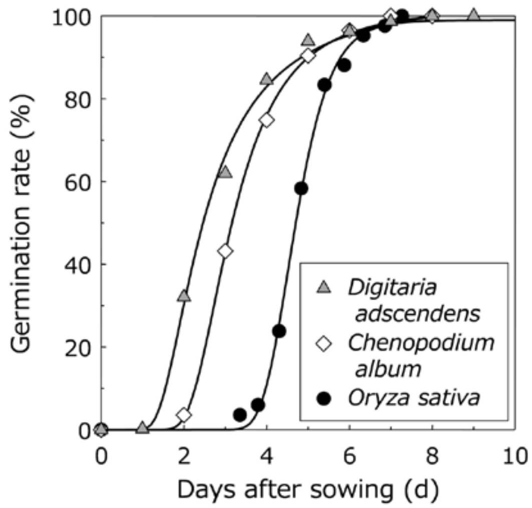

Fig. 4. Fitting of reach time equations for the trends in germination rate of different plant species
For the germination time of each plant species, the measured values are shown as symbols and the calculated values are shown as a curve fitted with a different reach time equation. The quartile skewness (Sq) of the curves and the variation coefficient (CV) of $A{\cdot}c$ for germination were calculated as follows: *Digitaria adscendens*, $Sq = 0.341$, CV = 0.506, *Chenopodium album*, $Sq = 0.212$, CV = 0.314, and *Oryza sativa*, $Sq = 0.0953$, CV = 0.141.

Discussion

The reach time equation is derived from the theoretical Arrhenius equation, and is based on the reasonable assumption that the concentration factor is normally distributed. Hence, the reach time equation can also be considered theoretical.

Because the Arrhenius equation incorporates a temperature parameter, the derived reach time equation similarly contains a temperature parameter and can be used in analyses including the effect of temperature. If this temperature parameter is not required, the number of equation parameters can be reduced from five to three, the same as the number of parameters in the logistic equation, and one less than that in the Richards equation, which can be used to represent a skewed distribution. The reach time equation has the advantage in that skewed distribution can be represented by as few as three parameters.

When the equation proposed by Shibuya & Hayashi (1984) is supplemented with the parameter of the final rate, the number of parameters is three, as in the reach time equation. However, the former equation cannot describe small differences in the shape of the distributions owing to the number of fluctuations, which is the parameter determining the shape of the distribution, the values of which must be integers for factorial calculations. Thus, the reach time equation can be applied more readily than the equation described by Shibuya & Hayashi (1984).

In both these equations, the shape of the distribution is such that the anterior portion of the curve is contracted, the posterior portion is extended, and tails off along the time axis relative to the median of the distribution, which tends to reflect the distribution patterns of empirically observed population processes. Shibuya & Hayashi (1984) explained this as an increase in the number of fluctuations until something occurred. The reach time equation indicates that the pattern is attributable to variation in the concentration factor. Essentially, in the case of germination, it is considered that the variation in the concentration factor is a consequence of variations occurring before sowing, although it could include the variation occurring during the period from sowing to germination. Thus, the shape of the distribution can be assumed to reflect the variation (or fluctuations) occurring before seeding and up to germination. This is interesting because it explains the tailing-off of the distribution curve with increasing time. This could also explain why the time distribution to reach a certain stage is often fitted using the logistic equation, although the distribution is inherently skewed. This could be because they are often assessed using highly uniform populations that are unlikely to produce a skewed distribution. Consequently, in such cases, it is preferable to use the reach time equation, which can be explained theoretically.

The skewness of the distribution is associated with the variability of the concentration factor, and the degree of skewness depends exclusively on, and is proportional to the variation coefficient of the concentration factor (Eq. 25). Moreover, this indicates that the shape of the distribution of the reach times is determined solely by the variation coefficient of the concentration factor in the population. It is not only qualitatively understandable that skewness increases with a variation in concentration, but also possible to estimate the skewness of the distribution from the variation coefficient of the concentration factor and, conversely, the variation coefficient from the skewness, although in this case, sufficient data are required.

For example of applying reach time analysis, in this study, the variation coefficient of the concentration factor associated with germination was estimated from the distribution of the germination times of crops and weeds (Fig. 4). However, it is not yet clear whether this actually corresponds to the variation coefficient in the amounts of substrates or enzymes in the seeds or the weight of the seeds, and even if it does, what it is useful for. Nevertheless, it would be interesting to obtain parameters representing such population characteristics, and it is anticipated that this approach will be useful in analyzing the reach time characteristics of populations. Although seed germination data were used for trial assessment in this study, only the Arrhenius equation was used to derive the reach time equation, with the requirement of a normal distribution. Thus, analyses using this equation would not simply be limited to germination, but could be used more generally to include the reach time of other growth stages, including the reach time of any stage for inanimate objects such as nutrients supplied from fertilizers and machinery failure. Not only for homogeneous populations, but also for the population which similar but different populations are combined, the overall characteristics can be captured by treating the synthesized population as a single broad population. For example, the reach time concept could be applied to collectively determine the supply of nutrients from organic matter with varying degrees of decomposition. Furthermore, there may be cases in which it is desirable to use a distribution other than a normal distribution. In this regard, although it is not entirely clear which types of phenomena could be used, it is hoped that in the future, this method will be more widely assessed for its applicability in analyzing a diverse range of phenomena.

Conclusion

Based on the theoretical Arrhenius equation, assuming a normal distribution, the reach time equation describes the time distribution required to reach a certain growth stage. Because it is a theoretical equation, it can be used for populations from a theoretical perspective, including for simple fitting, as it is based on the same or fewer parameters as substitutively used for growth equations. The equation reveals that skewness, which determines the shape of the distribution, is dependent on the variation in the concentration factor, and is specifically proportional to its variation coefficient. The equation is anticipated to be of practical utility in analyzing a diverse range of population processes, including germination, and encompassing other growth stages and inorganic changes.

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